GOSTEAM Hands-on Activity Template *(Classroom-Formal)*

Title:

Distances with vector algebra

Short Description (Max 500 words):

Students determine distances in 3-dimensional coordinate system with the use of advanced technology

Keywords (Up to 5):

Vector algebra, distances, mathematics, GeoGebra

Information about the Implementation

Age and language of the students:

Language:

Age:	9-12	12-15	15-18	18+
	☐	☐	■	☐

Number of Lessons – Duration (per lesson):

Number of Lessons: 1-2

Duration per Lesson: full time

Subjects:

For which subject(s) the activity is usable, is it an interdisciplinary activity?

Science ☐

Physics ☐ Chemistry ☐ Biology ☐ Geosciences ☐ Environmental ☐ Other ☐

Technology ☐

Engineering ☐

Arts ☐

Mathematics ■

Information about the Scenario

Curriculum and country:

Link of the current activity to the curriculum:

Country:	Austria	Class:	AHS	Grade:	10

Topic: vector algebra (use in applications)

Objectives (Max 100 words):

Description of the learning objectives

Students determine distances in 3-dimensional coordinate system with the use of advanced technology

Materials (Max 100 words):

Which resources and materials (software, hardware) are needed?

GeoGebra 6 (language English is available in the settings); Basic knowledge about vectors, lines und plains in three dimensions.

Spatial concepts, skills and abilities:

Which spatial concepts and skills are covered by the activity?

Spatial concepts:

Primitives: Identity/Name ☐ Location ☐ Space/Time ■

Simple: Distance ■ Direction ■ Connectivity ☐ Movement ■ Boundary ☐ Shape/Area ☐ Adjacency ☐

Difficult: Overlay ☐ Buffer ☐ Topology ☐ Coordinate ■ Map ☐ Scale ■ Shortest Path ☐ Navigation ☐ Surface ☐ Slope/Gradient ☐ Aspect ☐ Contour ☐

Complex: Interpolation ☐ Map Projection ☐ Spatial Dependency ☐

Other: [ ]

Spatial skills:

- ☐ Map literacy
- ☐ Navigation/orientation
- ■ Estimating distances and directions
- ☐ Recognizing and understanding patterns/Understand and identify models of spatial organization
- ☐ Select an ideal location based on the given spatial features
- ☐ Visualization
- ☐ Understand and identify spatial correlations/ dependencies
- ☐ Categorize spatial entities/ geographic features and identify hierarchies
- ☐ Compare spatial entities and draw analogies among them
- ☐ Identify/determine connections/relations
- ■ Understanding scale in space and time
- ☐ Delineation of spatial regions/ zones based on given features/ properties

Short Description

Navigation/orientation: Finding one's way in unfamiliar environments, interpreting and giving walking and driving directions.
Estimating distances and directions: Measure paths, weighted distances, angles.
Map literacy: Using, interpreting/understanding, learning from, and communicating acquired spatial knowledge from maps, comprehension of geographic features represented as points, lines, or polygons.
Recognizing and understanding patterns/Understand and identify models of spatial organization. Delineation of spatial regions/zones based on given features/properties: Regionalization processes, pattern recognition and clustering identification in the 2d and/or the 3d world.
Select an ideal location based on the given spatial features: Single or multi-criteria siting and optimal areas identification.
Visualization: Visualizing spatial entities from written/oral verbal descriptions, from their 2d or graphical representations or through mental transformations; such as axis rotation or perspective taking.
Understand and identify spatial correlations/ dependencies: The ability to realize, identify and explain patterns, clusters and relevant spatial dependencies.
Categorize spatial entities/geographic features and identify hierarchies: Identify the hierarchical form of data and gradients between spatial entities.
Compare spatial entities and draw analogies among them: Calculate and compare different geometric objects' shapes, area and, boundaries.
Identify/determine connections/relations: The ability to identify links and common characteristics among spatial entities and between humans and spatial entities.
Understanding scale in space and time: The understanding of changes/transitions through space and time for different spatio-temporal scales.

Geospatial concepts and spatial abilities documentation (see Section 3.2):
http://www.gosteam.eu/wp-content/uploads/2021/05/GOSTEAM_IO1_A1_final.pdf

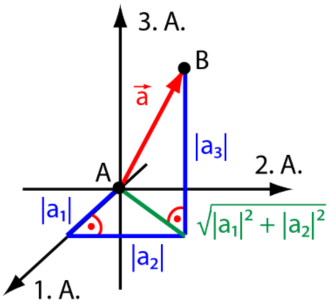
Description of the activity in detail

Classroom & Online activities

Engage

Imagine we have two points, which are moving in space, e. g. two airplanes or two ships. If they are moving straight forward, their motion can be described as a line. How it is possible to determine the shortest distance between these objects?

For the start, we want to determine the distance between two unmoving points in the 3-dimensional coordinate system. As we can choose the orientation of the system, we set one point A in the origin. How can the distance to point B be calculated? Do so for $B = (1,4,7)$.



Solution: We draw the vector $\vec{a} = \overrightarrow{AB}$ and find his length. As shown on the right side, the length is

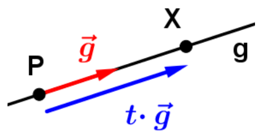
$$|\vec{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} = \sqrt{1^2 + 4^2 + 7^2} = \sqrt{66} \approx 8{,}1$$

Explain

We know want to find distances between points, lines and plains. It is not very easy to calculate such distances by hand and it takes a lot of time (see the elaboration sector). Therefore, we want to visualize the point, lines and plains and determine the distances with the use of technology.

We have to remember some things:

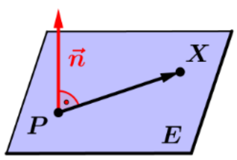
A **line** $\boldsymbol{g}$ with direction $\vec{g}$ and any point $P \in g$ can be described through $g: X = P + t \cdot \vec{g}$. There, $t \in \mathbb{R}$ ist a parameter.



In components, we write

$$g: X = (p_1, p_2, p_3) + t \cdot (g_1, g_2, g_3)\,.$$

A **plain** $\mathbf{E}$ with normal vector $\vec{n} \neq \vec{o}$ and any point $P \in E$ can be described through $\vec{n} \cdot X = \vec{n} \cdot P = const.$



In components, we write

$$n_1 \cdot x + n_2 \cdot y + n_3 \cdot z = const.$$

Explore

Before we start, we have to know how to use Geogebra to visualize objects und determine distances.

Technology information: Open GeoGebra 6. Click on the settings and choose *Perspectives* and then *3D Graphics*. On the left side, there is the algebra window, where we can type in any function. On the right side is the 3D graphic window, where the graph of the function will be shown.

There are several options to type in lines and plains. You can type them in the algebra window in the form $(p_1, p_2, p_3) + t \cdot (g_1, g_2, g_3)$ for a line or $n_1 x + n_2 y + n_3 z = c$ for a plain. If needed you can also use the toolbox at the top to draw lines or plains.

To determine the distance, use the tool *Distance or Length* in the toolbox (to find under the angle). Click in the 3D graphic window one object after the other, from which you want to find the distance between. Unfortunately, you cannot choose plains with this tool. For them, you can use the command Distance(<point>, <object>) in the algebra window.

Task: Determine the distances between the following objects. Make use of moving the 3D Graphic window to get a clear sight, how the objects are placed in the coordinate system.

(1) $P = (15, -2, 7),\ \ E: 6x - 3y + 2z = 12$

(2) $E_1 =$ ABC with $A = (-1,4,6), B = (1,3,7), C = (5,2,12),\ \ E_2: 2x + 3y - z = 3$

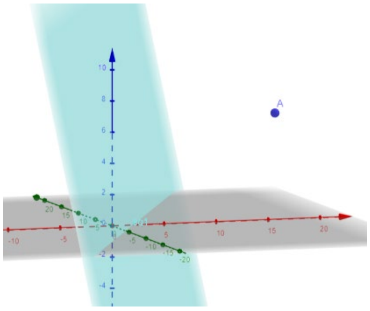
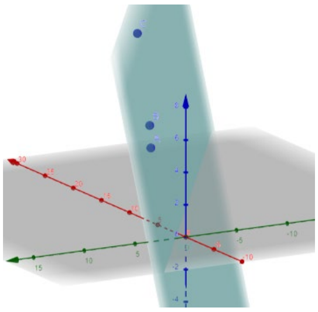
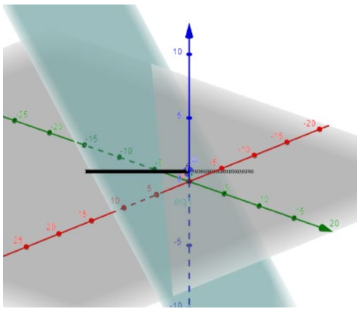
(3) $g: X = (-2,3,-5) + t \cdot (3,2,1),\ \ E: x - y - z = 4$

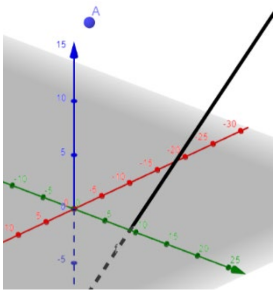
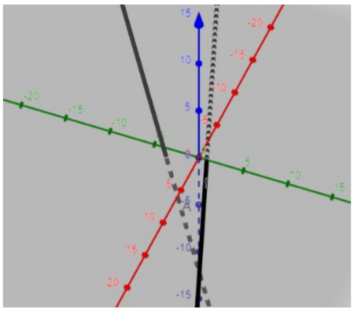
(4) $P = (5,7,20),\ \ g: X = (3,5 - 6) + t \cdot (-5,4,7)$

(5) $g: X = (3,2,1) + s \cdot (8,3,3),\ \ h: X = (1,-1,7) + t \cdot (0,-1,3)$

Solution:

(1) $d = 14$ (2) $d = 0{,}27$ (3) $d = 2{,}31$ (4) $d = 18$ (5) $d = 4$

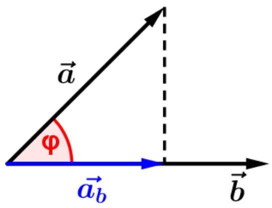



Elaborate

Where is the shortest distance exactly to draw? And how can the distance be exactly calculated?

Solution: The shortest distance is the length of a line from a point to the object, in which the line has a right angle to the object.

For the calculation of the distance between a point and a plain, we can use the normal projection and the scalar product. As you probably know, is



$$\left|\vec{a}\cdot\vec{b}\right| = |\vec{a}|\cdot\left|\vec{b}\right|\cdot|\cos\varphi| = |\vec{a}_b|\cdot\left|\vec{b}\right|$$

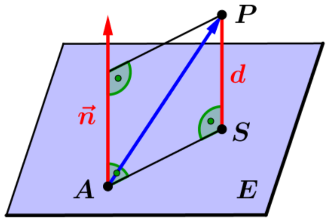
because

$$|\cos\varphi| = \frac{|\vec{a}_b|}{|\vec{a}|}$$

We can find the distance between a point and plain, when we compare the graphic of the normal projection with the following graphic.

Distance formula of Hesse: Let $P \in \mathbb{R}^3$, E a plain in $\mathbb{R}^3$ with normal vector $\vec{n}$ and A any point at E. The distance d between the point P and the plain E is given through:

$$d = \frac{\left|\overrightarrow{AP}\cdot\vec{n}\right|}{|\vec{n}|}$$



The formula is named after the german mathematician **Ludwig Otto Hesse** $(1811 - 1874)$. The formula can be also used to calculate the distance between two parallel plains and between a plain and a parallel line, when you choose any point P of one of the plains respectively the line.

Task: Calculate the distance d for the following situations without the use of technology:

(1) $P = (2,-7,18),\ \ E: 2x - y - 2z = 2$

(2) $E: y - z = 3,\ \ g: (1,0,0) + t\cdot(2,3,3)$

Solution:

(1)

$$\vec{n} = (2,-1,-2),\ \ A = (1,0,0),\ \ P = (2,-7,18)$$

$$\overrightarrow{AP} = P - A = (1,-7,18)$$

$$\left|\overrightarrow{AP}\cdot\vec{n}\right| = |(2,-1,-2)\cdot(1,-7,18)| = |2 + 7 - 36| = |-27| = 27$$

$$|\vec{n}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$d = \frac{27}{3} = 9$$

(2)

$$\vec{n} = (0,1,-1),\ \ A = (0,3,0),\ \ P = (1,0,0)$$

$$\overrightarrow{AP} = P - A = (1,-3,0)$$

$$\left|\overrightarrow{AP}\cdot\vec{n}\right| = |(1,-3,0)\cdot(0,1,-1)| = |0 - 3 + 0| = |-3| = 3$$

$$|\vec{n}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$$

$$d = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

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References (if any):

Assessment (if any):